ON THE NILCLOSURE IN A CATEGORY OF UNSTABLE MODULES

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Abstract

In this paper, we investigate nilclosure of $F(2)^{\otimes n} / \Sigma_n$ in the quotient category $\mathcal{U} / \mathcal{Nil}'$, where $\mathcal{U}'$ is the category of evenly graded unstable modules, $\mathcal{Nil}'$ is the category of evenly graded nilpotent unstable modules, and $\Sigma_n$ is a symmetric group.

1. Introduction and Main Results

Let $p$ be an odd prime number and $\mathcal{A}$ be the mod-$p$ Steenrod algebra. Let $\mathcal{U}$ be the category of unstable modules over mod-$p$ Steenrod algebra $\mathcal{A}$. In [3], Gabriel constructed the quotient category. Later, Franjou and Schwartz [2] have introduced $\mathcal{U}/\mathcal{Nil}$ the quotient category of unstable modules by nilpotent modules. Henn et al. [4] have proved that $\mathcal{U}/\mathcal{Nil}$ is equivalent to representation theoretic category $\mathcal{F}_{w}$, where category $\mathcal{F}_{w}$

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is the category of analytic functors. Kuhn [7] has given more general description of category $\mathcal{F}_w$ called there 'category of locally finite functors'.

In this paper, we are restricting our attention to full subcategory $\mathcal{U}'$ of $\mathcal{U}$ consisting of unstable modules, which are nontrivial only in even degree. So $F'(2n)$ is free unstable modules over the subalgebra $A_p$ of $A$ generated by reduced power $P^i, i > 0$. $F'(2)^{\otimes n}$ is the unstable modules because tensor product of unstable modules with diagonal action is unstable. The symmetric group $\Sigma_n$ acts on $F'(2)^{\otimes n}$ by permutation of coordinates. Hence, we have

$$F'(2n) \simeq (F'(2)^{\otimes n})^{\Sigma_n},$$

(see [8]).

In the category $\mathcal{F}_w$, there exists naturally arising algebra

$$M_{*,*} = \text{Hom}_{\mathcal{F}_w}(S_*, S^*),$$

where $S_*$ and $S^*$ denote the graded symmetric invariants and coinvariants, respectively.

In [7], Kuhn has shown that $M_{*,*}$ is isomorphic to a polynomial algebra generated by the elements $u_{(1, p^k)}$ and $u_{(p^k, 1)}$, where $u_{(1, p^k)}$ has bidegree $(1, p^k)$ and $u_{(p^k, 1)}$ has bidegree $(p^k, 1)$. Hence, this allows us to compute nilclosure of $(F'(2)^{\otimes n})^{\Sigma_n}$. Using the equivalence,

$$\mathcal{F}_w \simeq \mathcal{U}/\text{Nil}',$$

we have the isomorphism

$$M_{*,j} = \mathcal{N}(F'(2)^{\otimes j})/\Sigma_j.$$
2. Preliminaries

Definition 2.1. An \( A \)-module \( M \) in the category \( \mathcal{U}' \) is called unstable, if \( P^i(x) = 0 \) for \( 2i > |x| \) and \( x \in H^*(X) \), where \( |x| \) denotes the degree of \( x \).

Let \( \Phi' \) denote a functor from \( \mathcal{U}' \) to itself. Given \( M \in \mathcal{U}' \) define \( \Phi'M \) by

\[
(\Phi'M)^{2m} = \begin{cases}
\frac{2m}{p}, & \text{if } 2m \equiv 0 \mod 2p, \\
\frac{2m-2+1}{p}, & \text{if } 2m \equiv 2 \mod 2p, \\
0, & \text{otherwise}.
\end{cases}
\]

The action of the Steenrod algebra is given by

\[
P^i\Phi'(x) = \begin{cases}
\Phi'^{i/p}(x), & \text{if } i \equiv 0 \mod p, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\beta \Phi'(x) = 0,
\]

where \( \Phi'(x) \) denotes the element of \( \Phi'M \) corresponding to \( x \) in \( M \). The map

\[
\lambda'_M : \Phi'M \longrightarrow M
\]

is defined by \( \lambda'_M(\Phi'x) = P^m(x) = P_0(x) \), where the degree of \( x \) is \( 2m \).

Define an \( A \)-linear map

\[
P_{2i} : M^{2m} \longrightarrow M^{2mp-2i(p-1)},
\]

by \( P_{2i}(x) = P^{m-i}(x) \).
We denote by $\Sigma' : U' \longrightarrow U'$ the suspension functor, by $\tilde{\Phi}'$ (resp., $\tilde{\Sigma}'$) the right adjoint of $\Phi'$ (resp., $\Sigma'$) and $R^i \tilde{\Phi}'$ (resp., $R^i \tilde{\Sigma}'$) the derived functor of $\tilde{\Phi}'$ (resp., $\tilde{\Sigma}'$). Let
\[\Sigma' i_{2m} : F'(2(m + 1)) \longrightarrow \Sigma' F'(2m)\]
be the map, which sends $i_{2(m+1)}$ to $\Sigma' i_{2m}$.

It is easy to check that tensor product of two unstable modules equipped with the diagonal action is unstable. So consider the unstable module $F'(2)\otimes^n$; it is of dimension 2 in degree $n$. Therefore, up to a scaler multiple, there is one nontrivial map
\[h : F'(2n) \longrightarrow F'(2)\otimes^n,\]
defined by $h(i_{2n}) = y \otimes y \otimes \cdots \otimes y$. The symmetric group $\Sigma_n$ acts on $F'(2)\otimes^n$ by permuting factor, i.e.,
\[\sigma(y^{p_{a_1}} \otimes y^{p_{a_2}} \otimes \cdots \otimes y^{p_{a_n}}) = y^{p_{a_{\sigma^{-1}(1)}}} \otimes \cdots \otimes y^{p_{a_{\sigma^{-1}(n)}}}.\]

The action $\Sigma_n$ of $F'(2)\otimes^n$ commutes with Steenrod algebra action and therefore the range of $h$ is contained in the invariants $(F'(2)\otimes^n)\Sigma^n$. So, we have the following lemma:

**Lemma 2.2.** The map $h : F'(2n) \longrightarrow (F'(2)\otimes^n)\Sigma^n$ is an isomorphism.

**Definition 2.3.** Let $M$ be an unstable module in $U'$. An element $x$ of an unstable module $M$ is called nilpotent, if
\[P^{k\frac{h}{2}} P^{k-1\frac{h}{2}} \cdots P^{\frac{h}{2}}(x) = 0,\]
for some positive integer $k$. An unstable module $M$ is said to be nilpotent, if every element of $M$ is nilpotent.
Now, we can consider the quotient category \( \mathcal{U}'/\text{Nil}' \). If \( r' : \mathcal{U}' \to \mathcal{U}'/\text{Nil}' \) is the projection functor, then there exists a right adjoint \( s' \) to \( r' \), i.e., a functor \( s' : \mathcal{U}'/\text{Nil}' \to \mathcal{U}' \) such that

\[
\text{Hom}_{\mathcal{U}'/\text{Nil}'}(r'M, N) = \text{Hom}_{\mathcal{U}'}(M, s'N).
\]

**Definition 2.4.** Let \( \text{Nil}'(M) \) denote the submodule of nilpotent elements of \( M \).

1. An unstable module \( M \) is said to be **reduced**, if \( \text{Nil}'(M) = 0 \).
2. An unstable module \( M \) is called **\( \text{Nil}' \)-closed**, if the unit of adjoint \( \varepsilon : M \to s'r'M \) is an isomorphism.

From [1, Proposition 2.1], we have the following lemma:

**Lemma 2.5.** An unstable module \( M \) is **\( \text{Nil}' \)-closed**, if and only if \( \text{Ext}_{\mathcal{U}'}^i(N, M) = 0 \) for any nilpotent module \( N \) and \( i = 0, 1 \).

As a direct consequence of Lemma 2.5, we note that a product of \( \text{Nil}' \)-closed unstable \( \mathcal{A}_p \)-modules is \( \text{Nil}' \)-closed and that the kernel of a map between \( \text{Nil}' \)-closed unstable \( \mathcal{A}_p \)-modules is \( \text{Nil}' \)-closed. Therefore, any inverse limit of \( \text{Nil}' \)-closed unstable \( \mathcal{A}_p \)-modules is still \( \text{Nil}' \)-closed.

Let \( \mathcal{N}(M) \) denote \( s'r'M \). So, it has the following universal property: Given an \( \mathcal{A}_p \)-linear map \( g' : M \to N \), where \( N \) is \( \text{Nil}' \)-closed, there exists a unique \( \mathcal{A}_p \)-linear map \( \mathcal{N}(g') \) making the following diagram commutative:

\[
\begin{array}{ccc}
M & \longrightarrow & \mathcal{N}(M) \\
\downarrow f & & \downarrow \mathcal{N}(g') \\
N & = & N.
\end{array}
\]

The following result immediately follows from [9, Theorem 6.3.3] and [9, Theorem 6.3.4].
Proposition 2.6. (1) An \( A_p \)-module \( M \in U' \) is \( \text{Nil}' \)-closed, if and only if the adjoint to \( \lambda' \)
\[ \tilde{\lambda}' : M \to \tilde{\Phi}'M \]
is an isomorphism.

(2) The functor \( N : \text{colim}_k \tilde{\Phi}'^k M \) and \( N(M) \) are naturally equivalent.

Let \( E_\infty \) and \( E \) be the category of \( \mathbb{F}_p \)-vector space and the category of finite dimensional vector space, respectively. Let \( \mathcal{F} \) denote the category of covariant functors and natural transformation from \( E \) to \( E_\infty \).

Definition 2.7. (1) A sequence \( F \to G \to H \) is exact if for any elementary abelian \( p \)-group \( V \), the sequence \( F(V) \to G(V) \to H(V) \) is exact.

(2) A functor \( F \in \mathcal{F} \) is simple, if the only subobjects of \( F \) are the 0 and \( F \) itself.

(3) A functor \( F \in \mathcal{F} \) is finite, if it has a finite composition series with simple subquotients.

(4) A functor \( F \in \mathcal{F} \) is locally finite, if it is the union of its finite subfunctors.

Let \( \mathcal{F}_w \) denote the full subcategory of locally finite functor. The definition of \( \mathcal{F}_w \) as in (4) is due to Kuhn and it is different from the original definition given in [4]. However, it is also shown in [6] that the two definitions are equivalent.

We now define a functor \( f' : \mathcal{U}' \to \mathcal{F} \) as follows: The functor \( f'(M) : E \to E_\infty \) associated to an unstable \( A_p \)-module \( N \) is given by the formula
\[ f'(M)(V) = (\text{Hom}_{\mathcal{F}}(M, H^*V))^\prime, \]
where \((\text{Hom}_{\mathcal{U}}(M, H^*V))'\) denotes the continuous dual of profinite \(\mathbb{F}_p\)-vector space

\[
\text{Hom}_{\mathcal{U}}(M, H^*V).
\]

The profinite structure of \(\text{Hom}_{\mathcal{U}}(M, H^*V)\) comes from the fact that this \(\mathbb{F}_p\)-vector space is the filtered inverse limit of the finite dimensional \(\mathbb{F}_p\)-vector space \(\text{Hom}_{\mathcal{U}}(M_\alpha, H^*V), M_\alpha\) running through the set of sub-\(A_p\)-modules of \(M\) finitely generated over \(A\); an element of \((\text{Hom}_{\mathcal{U}}(M, H^*V))'\) is a continuous linear map from \(\text{Hom}_{\mathcal{U}}(M, H^*V)\) equipped with this profinite topology to \(\mathbb{F}_p\) equipped with its discrete topology.

Let \(M\) be an unstable \(A_p\)-module and \(E\) be \(\mathbb{F}_p\)-vector space. Let us assume first that \(M\) is finitely generated over \(A_p\) and \(E\) finite dimensional. It is clear that in this case that one has a natural isomorphism

\[
\text{Hom}_{\mathcal{U}}(M, E \otimes H^*(V)) \simeq \text{Hom}_{\mathcal{E}_\infty}(f'(M)(V), E)
\]

(in this formula \(E\) is viewed as an unstable \(A\)-module concentrated in degree zero). Taking filtered direct limits, first over \(E\) and then over \(M\), shows that this isomorphism holds without assumptions on \(M\) and \(E\). Thus, the functor \(\mathcal{U}' \longrightarrow \mathcal{E}_\infty, M \longmapsto f'(M)(V)\) is left adjoint to functor \(\mathcal{E}_\infty \longrightarrow \mathcal{U}', E \longmapsto E \otimes H^*(V)\). This shows that \(f'(M)(V)\) identifies with the vector space \((T'_V(M))_0^0\) of degree zero elements in the unstable \(A\)-module \(T'_V(M), T'_V : \mathcal{U}' \longrightarrow \mathcal{U}'\) denoting the functor left adjoint to the functor \(\mathcal{U}' \longrightarrow \mathcal{U}', N \longmapsto H^*V \otimes N\). So, we can state the properties of the functor \(f'\), which is analogue of the result in [8].
**Proposition 2.8.** (1) The functor $f'$ is exact.

(2) The functor $f'$ commutes with the tensor products.

So $f'$ is zero on $\text{Nil}'$ and hence $f'$ admits a factorization:

$$
\begin{array}{ccc}
\mathcal{U}' & \xrightarrow{f'} & \mathcal{F} \\
\downarrow{r'} & & \uparrow{\bar{f}} \\
\mathcal{U}'/\text{Nil}' & \cong & \mathcal{U}'/\text{Nil}'.
\end{array}
$$

By [4, Theorem 1.6.2], we immediately have the following result:

**Proposition 2.9.** The functor $\bar{f}'$ induces an equivalence of abelian category $\mathcal{U}'/\text{Nil}' \longrightarrow \mathcal{F}_w$.

The functor $m': \mathcal{F}_w \longrightarrow \mathcal{U}'$, which is corresponding to $s': \mathcal{U}'/\text{Nil}' \longrightarrow \mathcal{U}'$ is a right adjoint of $f'$. So $m'$ is given as follows:

$$m'(G)^n = \text{Hom}_\mathcal{F}(H^n, G),$$

and therefore, we have

$$m'f'(M) = N(M).$$

### 3. Main Results

Let $(k, \alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$, be a sequence of nonnegative integers. Define an equivalence relation on the set of such sequence by

$$(k, \alpha_1, \alpha_2, \ldots, \alpha_n) \sim (k-1, \alpha_1-1, \ldots, \alpha_n-1),$$

if $\alpha_1 > 0$ and $k > 0$. Then we can call $(k, \alpha_1, \ldots, \alpha_n)$ irreducible, if $k = 0$ or $\alpha_1 = 0$.

**Theorem 3.1.** Let $S$ be the set of irreducible sequence. Let $B$ be subset of $S$ with the following property: An irreducible sequence $(k, \alpha_1, \ldots, \alpha_n)$ is in $B$, if
(1) $k = 0$, or

(2) $k > 0$ and the number of $\alpha_i$ equal to $j$ is divisible by $p^{k-j}$, where $0 \leq j \leq k - 1$.

Then Nilclosure of $F'(2)^{\otimes n} / \Sigma_n$ has a $\mathbb{F}_p$-basis corresponding to subset $B$ of $S$.

**Proof.** Note that a sequence $(k, \alpha_1, \ldots, \alpha_n)$ will correspond to the element $x \in F'(2)^{\otimes n} / \Sigma_n$ such that

$$p^{k|}(x)p^{k-1|}(x) \cdots p^{|}(x) = [y^{p\alpha_1} \otimes y^{p\alpha_2} \otimes \cdots \otimes y^{p\alpha_n}],$$

where $x$ has even degree and $[\cdots]$ denotes the class of the monomial $y^{p\alpha_1} \otimes y^{p\alpha_2} \otimes \cdots \otimes y^{p\alpha_n}$ in $F'(2)^{\otimes n} / \Sigma_n$. Then, the module $F'(2)^{\otimes n} / \Sigma_n$ has a basis corresponding to the orbits of monomials

$$y^{p\alpha_1} \otimes y^{p\alpha_2} \otimes \cdots \otimes y^{p\alpha_n},$$

where $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_n$. It is clear that these monomials correspond to the irreducible sequences with $k = 0$. First, let us attach Steenrod operations to $F'(2)^{\otimes n} / \Sigma_n$, i.e., the element $x \in F'(2)^{\otimes n} / \Sigma_n$ such that

$$p^{|}(x) = [y^{p\alpha_1} \otimes y^{p\alpha_2} \otimes \cdots \otimes y^{p\alpha_n}].$$

Then we get the module $M_1$ in this way. Using the Proposition 2.6, we have the following:

$$F'(2)^{\otimes n} / \Sigma_n \xrightarrow{\mathcal{N}(f)} \mathcal{N}(F'(2)^{\otimes n} / \Sigma_n).$$

Let $[y^{p\alpha_1} \otimes y^{p\alpha_2} \otimes \cdots \otimes y^{p\alpha_n}]$ be a class in $F'(2)^{\otimes n} / \Sigma_n$ and $\alpha_1 > 0$. Then, we will have
\[ P^{|k|}(x) = \left[ y^{p_{\alpha_1}} \otimes y^{p_{\alpha_2}} \otimes \cdots \otimes y^{p_{\alpha_n}} \right], \]

where \( x = y^{p_{\alpha_1}} \otimes y^{p_{\alpha_2}} \otimes \cdots \otimes y^{p_{\alpha_n}} \).

If \( \alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_k = 0, \alpha_{k+1} > 0 \), and \( k \) is nonzero in mod-\( p \), then the monomial class

\[ [y \otimes y \otimes \cdots \otimes y^{p_{\alpha_{k+1}}} \otimes \cdots \otimes y^{p_{\alpha_n}}], \]

does not admit a Steenrod operation because of degree.

Let \( \alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_k = 0 \) and \( k \) is zero in mod-\( p \). Then consider the coaction map

\[ \lambda : M \longrightarrow M \otimes A^*, \]

where \((M \otimes A^*)^n = \prod_{k=i=n} M_k \otimes A^*_i\), is then given by

\[ \lambda(m) = \sum m_I \otimes \xi^I, \]

if and only if for any \( \theta \in A \), we have

\[ \theta m = \sum < \theta, \xi^I > m_I, \]

where \(< . . >\) denotes the standard Kronecker pairing between \( A \) and \( A^* \).

So using the property of \( \lambda \), we have

\[ \lambda(y_1 \otimes \cdots \otimes y_k \otimes y^{p_{\alpha_{k+1}}} \otimes \cdots \otimes y^{p_{\alpha_n}}) = \left( \sum_{i_k \geq 0} \right) \left( y^{p_{i_k}} \otimes \xi_{i_k} \right) \]

\[ \cdots \left( \sum_{i_k \geq 0} \right) \left( y^{p_{i_k+1}} \otimes \xi_{i_k+1} \right) \cdots \]

\[ \left( \sum_{i_n \geq 0} \right) \left( y^{p_{i_n}} \otimes \xi_{i_n} \right). \]
Let \( k = \sum_{j=1}^{r} l_j \) and \( \rho_i = \sum_{m=1}^{i} l_m \). Then we get

\[
\sum_{i_1, \ldots, i_r \geq 0 \atop i_{k+1}, \ldots, i_n \geq 0} (y_1 \otimes \cdots \otimes y_{p_1})^{p^i_1} \otimes \cdots \otimes (y_{p_1+1} \otimes \cdots \otimes y_{p_2})^{p^{i_2}} \\
\otimes \cdots \otimes (y_{p_{r-1}+1} \otimes \cdots \otimes y_{k})^{p^i_r} \otimes y^{p_{k+1} + i_{k+1}}_{k+1} \\
\otimes \cdots \otimes y^{p_{a_n+1+n}}_n \otimes \xi^{p^i_1}_{i_1} \cdots \xi^{p^i_r}_{i_r} \xi^{p_{k+1}}_{k+1} \cdots \xi^{p_{a_n}}_{a_n}. \quad (*)
\]

It is clear that terms of the form

\[
y^{p^i_r}_{i_r} \otimes y^{p^i_s}_{i_s} \otimes \xi_{r \neq s} + y^{p^i_r}_{i_r} \otimes y^{p^i_s}_{i_s} \otimes \xi_{r \neq s},
\]

for \( r \neq s \) are zero because of the action of \( \sum_n \). So terms of the form

\[
(y^{p^i_r}_{i_r} \otimes y^{p^i_s}_{i_s}) \otimes (y^{p^{i_1}}_{i_1} \otimes y^{p^{i_2}}_{i_2}) \otimes \xi^{p^i_r}_{i_r} \xi^{p^i_s}_{i_s} + (y^{p^i_r}_{i_r} \otimes y^{p^i_s}_{i_s}) \otimes (y^{p^i_r}_{i_r} \otimes y^{p^i_s}_{i_s}) \otimes \xi^{p^i_r}_{i_r} \xi^{p^i_s}_{i_s},
\]

for \( r \neq s \) are zero. So (*) holds by induction. We attach to \( F'(2)_{\otimes n} / \sum_n \) Steenrod operation of elements of the form

\[
y_1 \otimes y_2 \otimes \cdots \otimes y_k \otimes y^{p_{a+k+1}}_{k+1} \otimes \cdots \otimes y^{p_{a+n}}_n.
\]

The property of \( \lambda \) helps us to define the coaction map

\[
\lambda((y_1 \otimes \cdots \otimes y_k \otimes y^{p_{a+k+1}}_{k+1} \otimes \cdots \otimes y^{p_{a+n}}_n)^{p^{-1}}) \]

\[
= \sum_{i_1, \ldots, i_r \geq 0 \atop i_{k+1}, \ldots, i_n \geq 0} (y_1 \otimes \cdots \otimes y_{p_1})^{p^i_1-1} \otimes (y_{p_1+1} \otimes \cdots \otimes y_{p_2})^{p^{i_2-1}} \\
\otimes \cdots \otimes (y_{p_{r-1}+1} \otimes \cdots \otimes y_{p_k})^{p^{i_r-1}} \otimes \cdots \otimes y^{p_{a+k+1} + i_{k+1}-1}_{k+1} \\
\otimes \cdots \otimes \xi^{p^i_1-1}_{i_1} \cdots \xi^{p^i_r-1}_{i_r} \xi^{p_{a+k+1}-1}_{k+1} \cdots \xi^{p_{a+n}-1}_{a_n}. \quad (**)
\]
If any of the $i_1, \ldots, i_r$ in (** ) is zero, then
\[
(y_1 \otimes \cdots \otimes y_{p_1})^{p_{i_1}^{-1}} \otimes \cdots \otimes (y_{p_{r-1}+1} \otimes \cdots \otimes y_{p_k})^{p_{i_r}^{-1}} \otimes y_{k+1}^{n_{k+1}} \otimes \cdots \otimes y_n^{n_n}
\]
denotes
\[
((y_1 \otimes \cdots \otimes y_{p_1})^{p_{i_1}} \otimes \cdots \otimes (y_{p_{r-1}+1} \otimes \cdots \otimes y_{p_k})^{p_{i_r}} \otimes y_{k+1}^{n_{k+1}+1} \otimes \cdots \otimes y_n^{n_n+1})^{p^{-1}}.
\]

Next, we shall show that $M_1$ is an $A_p$-module, i.e., we need to verify that
\[
\theta_1(\theta_2 z) = (\theta_1 \theta_2) z
\]
for any $\theta_1, \theta_2 \in A_p$ and $z$ any of the attached elements. In terms of the coaction map $\lambda$, condition above can be translated into commutativity of the diagram
\[
\begin{array}{ccc}
M_1 \otimes A^*_p \otimes A^*_p & \overset{1 \otimes \Delta}{\longrightarrow} & M_1 \otimes A^* \\
\lambda \otimes 1 & & \lambda \\
\downarrow & & \downarrow \\
M_1 \otimes A^* & \overset{\lambda}{\longleftarrow} & M_1
\end{array}
\]
So, we show that the following equality:
\[
(\lambda \otimes 1) \lambda(z) = (1 \otimes \Delta) \lambda(z)
\]
holds, where $z = (y_1 \otimes y_2 \otimes \cdots \otimes y_k)^{p_{i_1}^{-1}} \otimes y_{k+1}^{n_{k+1}} \otimes \cdots \otimes y_n^{n_n}$.

First, let us find what the left side of equality is
\[
(\lambda \otimes 1) \lambda(z) = (\lambda \otimes 1) \sum_{i_1, \ldots, i_r \geq 0 \atop i_{k+1}, \ldots, i_n \geq 0} (y_1 \otimes \cdots \otimes y_{p_1}^{p_{i_1}^{-1}} \otimes \cdots \otimes (y_{p_{r-1}+1} \otimes \cdots \otimes y_{p_k}^{p_{i_r}^{-1}} \otimes y_{k+1}^{n_{k+1}+1} \otimes \cdots \otimes y_n^{n_n})
\]
\[
\otimes \cdots \otimes y_{p_{r-1}+1}^{p_{i_1}} \otimes \cdots \otimes y_{p_k}^{p_{i_r}^{-1}} \otimes \cdots \otimes y_n^{n_n+1})^{p^{-1}}.
\]
= \sum_{i_1, \ldots, i_r, j_1, \ldots, j_r \geq 0 \atop i_{k+1}, \ldots, i_n, j_{k+1}, \ldots, j_n \geq 0} \left( y_1 \otimes \cdots \otimes y_{p_1} \right) \rho^{i_1 + j_1 - 1} \otimes \cdots \otimes \left( y_{p_{r-1} + 1} \otimes \cdots \otimes y_{p_r} \right) \rho^{i_{r-1} + j_{r-1} - 1} \otimes y_{p_{r+1}} \rho^{a_{p_{r+1} + 1} + i_{k+1} + j_{k+1}} \otimes \cdots \otimes y_{p_{n+1}} \rho^{a_{p_{n+1} + 1} + i_n + j_n}

\otimes \cdots \otimes y_{p_{r-1} + 1} \otimes \cdots \otimes y_{p_r} \rho^{m_{r-1}} \otimes y_{p_{r+1}} \rho^{a_{p_{r+1} + 1} + m_{k+1}} \otimes \cdots \otimes y_{p_{n+1}} \rho^{a_{p_{n+1} + 1} + m_n}

The right side of equality is

(1 \otimes \Delta) \mathcal{h}(z) = \sum_{m_1, \ldots, m_n \geq 0 \atop m_{k+1}, \ldots, m_n \geq 0} \left( y_1 \otimes \cdots \otimes y_{p_1} \right) \rho^{m_1 - 1} \otimes \cdots \otimes \left( y_{p_{r-1} + 1} \otimes \cdots \otimes y_{p_r} \right) \rho^{m_{r-1} - 1} \otimes y_{p_{r+1}} \rho^{a_{p_{r+1} + 1} + m_{k+1}} \otimes \cdots \otimes y_{p_{n+1}} \rho^{a_{p_{n+1} + 1} + m_n}

On the other hand, we have

\Delta(\mathcal{z}_{\rho_{m_1}^{1}} \cdots \mathcal{z}_{\rho_{m_r}^{r}} \mathcal{z}_{\rho_{m_{k+1}}^{k+1}} \cdots \mathcal{z}_{\rho_{m_n}^{n}}) \notag

= \Delta(\mathcal{z}_{\rho_{m_1}^{1}}) \mathcal{z}_{\rho_{m_1}^{1}} \cdots \Delta(\mathcal{z}_{\rho_{m_r}^{r}}) \mathcal{z}_{\rho_{m_r}^{r}} \Delta(\mathcal{z}_{\rho_{m_{k+1}}^{k+1}}) \mathcal{z}_{\rho_{m_{k+1}}^{k+1}} \cdots \Delta(\mathcal{z}_{\rho_{m_n}^{n}}) \mathcal{z}_{\rho_{m_n}^{n}}

= \left( \sum_{i_1 + j_1 = m_1} \mathcal{z}_{\rho_{j_1}^{1}} \otimes \mathcal{z}_{i_1} \right) \mathcal{z}_{\rho_{j_1}^{1}} \cdots \left( \sum_{i_r + j_r = m_r} \mathcal{z}_{\rho_{j_r}^{r}} \otimes \mathcal{z}_{i_r} \right) \mathcal{z}_{\rho_{j_r}^{r}} \cdots \left( \sum_{i_n + j_n = m_n} \mathcal{z}_{\rho_{j_n}^{n}} \otimes \mathcal{z}_{i_n} \right) \mathcal{z}_{\rho_{j_n}^{n}}

= \sum_{i_1 + j_1 = m_1} \mathcal{z}_{\rho_{j_1}^{1}} \cdots \mathcal{z}_{\rho_{j_r}^{r}} \cdots \mathcal{z}_{\rho_{j_{k+1}}^{k+1}} \cdots \mathcal{z}_{\rho_{j_n}^{n}}

\otimes \mathcal{z}_{i_1} \cdots \mathcal{z}_{i_r} \cdots \mathcal{z}_{i_{k+1}} \cdots \mathcal{z}_{i_n}
When we compare both sides, the equality holds. Hence $M_1$ is unstable module. Similarly $M_2$ can be constructed. Therefore, the theorem follows by induction.

Let $M_{*,*}$ denote the morphism set in the category $\mathcal{F}$

$$\text{Hom}_{\mathcal{F}_w}(S_*, S^{*}),$$

where $S_*$ and $S^{*}$ are the graded symmetric invariants and coinvariants, respectively. So $M_{*,*}$ is a bigraded algebra with multiplication induced by the tensor product. Let $\alpha \in \text{Hom}_{\mathcal{F}_w}(S_i, S^{j})$ and $\beta \in \text{Hom}_{\mathcal{F}_w}(S_k, S^{l})$. Then $\alpha \cdot \beta$ is the composite

$$S_{i+k}(V) \xrightarrow{\vartheta} S_i(V) \otimes S_k(V) \xrightarrow{\alpha \otimes \beta} S^j(V^*) \otimes S^l(V^*) \xrightarrow{\rho} S^{j+l}(V^*),$$

where $\vartheta$ is a coproduct and $\rho$ is the cup product. Hence, $M_{*,*}$ is an $\mathcal{A}_p \times \mathcal{A}_p$-module with the $\mathcal{A}_p \times 1$-action on $M_{*,j}$ induced by the right action on homology and the $1 \times \mathcal{A}$-action on $M_{i,*}$ induced by the left action on cohomology. Using the equivalence between $\mathcal{U}'/\text{Nil}'$ and $\mathcal{F}_w$, we see that

$$M_{*,j} = \mathcal{N}(F'(2)^j / \Sigma_j).$$

Let

$$u_{(1, p^k)} = (y_1 \otimes \cdots \otimes y_{p^k})^{p^k} \in M_{1, p^k} \simeq \mathcal{N}(F'(2)^{p^k} / \Sigma_{p^k}),$$

and

$$u_{(p^k, 1)} = y^{p^k} \in M_{p^k, 1} \simeq \mathcal{N}(F'(2)^{p^k} \simeq F'(2)^{p^k}.$$
Theorem 3.2. The bigraded algebra $M_{*,*}$ is isomorphic to the polynomial algebra generated by the elements $u_{(1,p^k)}$ and $u_{(p^k,1)}$ of bidegree $(1, p^k)$ and $(p^k, 1)$, respectively.

Proof. Let $\alpha \in M_{i,j}$ and $\beta \in M_{k,l}$. Then $\alpha \beta \in M_{i+k,j+l}$, which is image of $\alpha \otimes \beta$ under tensor product

$$\mathcal{N}(F'(2)^m \otimes \Sigma_m) \to \mathcal{N}(F'(2)^m \otimes \Sigma_m).$$

By Theorem 3.1, any element in $\mathcal{N}(F'(2)^m / \Sigma_m)$ can be written as

$$(y_1 \otimes \cdots \otimes y_{p^n})^{p^{-k_1} + i_1} \otimes \cdots \otimes (y_{p^{n-1}} \otimes \cdots \otimes y_{p^n})^{p^{-k_n+i_n}} \otimes y_{p^{n+1}}^{i_{n+1}} \otimes \cdots \otimes y_{p^m}^{i_m} = u_{(1,p^k)}^{i_1} \cdots u_{(1,p^k)}^{i_n} u_{(p^k,1)}^{i_{n+1}} \cdots u_{(p^k,1)}^{i_m}.$$

This completes the proof. \qed

Let $K'(\cdot) = \mathbb{F}_p[x_i; i \in \mathbb{Z}]$ be a polynomial algebra on generators $x_k$ of bidegree $(1, p^k)$. Define $g_0 : M_{*,*} \longrightarrow K'(\cdot)$ by $g_0(u_{(p^k,1)}) = x_{p^k}$ and $g_0(u_{(1,p^k)}) = x_k$, where $u_{(p^k,1)}$ and $u_{(1,p^k)}$ have bidegrees $(p^k, 1)$ and $(1, p^k)$, respectively. So $g_0$ is an $A_p$-monomorphism.

Define $g_j : M_{*,p^j} \longrightarrow K'(\cdot)$ by $g_j(u_{(p^k,1)}) = x_{j-k} p^k$ and $g_j(u_{(1,p^k)}) = x_{k-j}$, $j \geq 1$. Since each map $g_j$ is mono, so is $g$. It is epi because by definition of $g_j$ every algebra generator is hit. So, we have just proved the following result:

Theorem 3.2. The map $g : \lim M_{*,*} \longrightarrow K'(\cdot)$ is an isomorphism.
References